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# CELLULARITY OF A LARGER CLASS OF DIAGRAM ALGEBRAS

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ABSTRACT. In this paper, we realize the algebra of $\mathbb{Z}_2$-relations, signed partition algebras and partition algebras as tabular algebras and prove the cellularity of these algebras using the method of [2]. Using the results of Graham and Lehrer in [1], we give the modular representations of the algebra of $\mathbb{Z}_2$-relations, signed partition algebras and partition algebras.

## 1. INTRODUCTION

The study of the algebra of $\mathbb{Z}_2$-relations and signed partition algebras are important because as they are subalgebras of partition algebras which arose naturally as potts model in statistical mechanics. In this paper, we establish the cellularity of the algebra of $\mathbb{Z}_2$-relations and signed partition algebras and hence deduce the modular representations of these algebras. The algebra of $\mathbb{Z}_2$-relations and signed partition algebras are different from the $\mathbb{Z}_2$-colored partition algebra introduced in [4] and Tanabe algebras introduced in [8] which are explained in section 3.

## 2. Preliminaries

**Definition 2.1.** *[[9]]*

*Let the group $\mathbb{Z}_2$ act on the set $X$. Then the action of $\mathbb{Z}_2$ on $X$ can be extended to an action of $\mathbb{Z}_2$ on $R(X)$, where $R(X)$ denote the set of all equivalence relations on $X$, given by*

$$g.d = \{(gp, gq) \mid (p,q) \in d\}$$

*where $d \in R(X)$ and $g \in \mathbb{Z}_2$. (It is easy to see that the relation $g.d$ is again an equivalence relation).*

*An equivalence relation $d$ on $X$ is said to be a $\mathbb{Z}_2$-stable equivalence relation if $p \sim q$ in $d$ implies that $gp \sim gq$ in $d$ for all $g$ in $\mathbb{Z}_2$. We denote $[k]$ for the set $\{1, 2, \ldots, k\}$. We shall only consider the case when $\mathbb{Z}_2$ acts freely on $X$; Let $X := [k] \times \mathbb{Z}_2$ and the action is defined by $g.(i, x) = (i, gx)$ for all $1 \leq i \leq k$. Let $R_k^{\mathbb{Z}_2}$ be the set of all $\mathbb{Z}_2$-stable equivalence relations on $X$.*

**Notation 2.2.** *$R_k^{\mathbb{Z}_2}$ denotes the set of all $\mathbb{Z}_2$-stable equivalence relations on $\{1, 2, \cdots, k\} \times \mathbb{Z}_2$.*

*Each element $d \in R_k^{\mathbb{Z}_2}$ can be represented as a simple graph on a row of $2k$ vertices.*

    (i) *The vertices $(1, e), (1, g), \cdots, (k, e), (k, g)$ are arranged from left to right in a single row.*

    (ii) *If $(i, g) \sim (j, g') \in R_k^{\mathbb{Z}_2}$ then $(i, g), (j, g')$ is joined by a line $\forall g, g' \in \mathbb{Z}_2$.*

*We say that the two graphs are equivalent if they give rise to the same set partition of the $2k$ vertices $\{(1, e), (1, g), \cdots, (k, e), (k, g)\}$. We may regard each element $d$ in $R_{2k}^{\mathbb{Z}_2}$ as a $2k$-partition diagram by arranging the $4k$ vertices $(i, g), i \in [2k], g \in \mathbb{Z}_2$ of $d$ in two rows in such a way that $(i, g)$ is in the top(bottom) row of $d$ if $1 \leq i \leq k (k + 1 \leq i \leq 2k)$ $\forall g \in \mathbb{Z}_2$ and put $(k + i, g) = (i', g), 1 \leq i \leq k$, for all $g \in \mathbb{Z}_2$ in the bottom row of $d$ and if $(i, g) \sim (j, g')$ then $(i, g), (j, g')$ is joined by a line $\forall g, g' \in \mathbb{Z}_2$.*

*The diagrams $d^+$ and $d^-$ are obtained from the diagram $d$ by restricting the vertex set to $\{(1, e), (1, g), \ldots, (k, e), (k, g)\}$ and $\{(1', e), (1', g), \ldots, (k', e), (k', g)\}$ respectively. The diagrams $d^+$ and $d^-$ are also $\mathbb{Z}_2$-stable equivalence relation and $d^+, d^- \in R_k^{\mathbb{Z}_2}$.*

## Definition 2.3. *[[9]]*

    *Let $d \in R_{2k}^{\mathbb{Z}_2}$. Then the equation*

$$R^d = \{(i, j) \mid \text{ there exists } g, h \in \mathbb{Z}_2 \text{ such that } ((i, g), (j, h)) \in d\}$$

    *defines an equivalence relation on $[2k]$.*

**Remark 2.4.** *[[9]] For $d \in R_{2k}^{\mathbb{Z}_2}$ and for every $\mathbb{Z}_2$-stable equivalence class or a connected component $C$ in $R^d$ there exists a unique subgroup denoted by $H_C^d$ where*

    (i) *$H_C^d = \{e\}$ if $(i, e) \nsim (i, g)$ $\forall i \in C$, $C$ is called an $\{e\}$-class or $\{e\}$-component and the $\{e\}$ component $C$ will always occur as a pair and*

    (ii) *$H_C^d = \mathbb{Z}_2$ if $(i, e) \sim (i, g)$ $\forall i \in C$, $C$ is called $\mathbb{Z}_2$-class or $\mathbb{Z}_2$-component and the number of vertices in the $\mathbb{Z}_2$-component $C$ will always be even.*

## Definition 2.5. *[[9]]*

    *The linear span of $R_{2k}^{\mathbb{Z}_2}$ is a subalgebra of $\mathbb{P}_{2k}(x)$. We denote this subalgebra by $A_k^{\mathbb{Z}_2}(x)$, called the **algebra of $\mathbb{Z}_2$-relations**.*

## Definition 2.6. *[[9]]*

    *Let $d$ be a $2k$-partition diagram. A connected component $C$ of $d$ which contains vertices in both the rows, is called a through class of $d$ and $\sharp^p(d)$ denotes the number of through classes of $d$, called propagating number. Any connected component $C$ of $d$ which contains vertices in only one row (either a top row or bottom row) is called a horizontal edge.*

*For $0 \leq 2s_1 + s_2 \leq 2k$, define $I_{2s_1+s_2}^{2k}$ to be the set of all $2k$-partition diagrams such that $\sharp^p(d) = 2s_1 + s_2$ for all $d \in I_{2s_1+s_2}^{2k}$.*

*i.e., $d$ has $s_1$ number of pairs of $\{e\}$ through classes and $s_2$ number of $\mathbb{Z}_2$ through classes.*

*Let $I_s$ be the linear space spanned by $\displaystyle\bigcup_{2s_1+s_2 \leq s} I_{2s_1+s_2}^{2k}$.*

**Definition 2.7.** ([5], *Definition 3.1.1*)

Let the **signed partition algebra** $\overrightarrow{A}_k^{\mathbb{Z}_2}(x)$ be the subalgebra of $\mathbb{P}_{2k}(x)$ generated by

$$H_1 = \quad\text{(diagram)}$$

$$F_i' = \quad\text{(diagram with } 2i \text{ labels)} \quad, 1 \le i \le k-1$$

$$G_i = \quad\text{(diagram)} \quad, 1 \le i \le k-1$$

$$F_i'' = \quad\text{(diagram with } 2i \text{ label)} \quad, 1 \le i \le k-1$$

$$F_j = \quad\text{(diagram with } 2j \text{ label)} \quad 1 \le j \le k$$

The subalgebra of the signed partition algebra generated by $F_i', G_i, F_i'', F_j, 1 \le i \le k-1, 1 \le j \le k$ is isomorphic on to the partition algebra $\mathbb{P}_{2k}(x^2)$.

**Definition 2.8.** ([5], *Definition 3.1.1*)

Let $d \in R_{2k}^{\mathbb{Z}_2}$. For $0 \le r = 2s_1 + s_2 \le 2k-1, 0 \le s_1, s_2 \le k-1$,

$$\widetilde{I}_{2s_1+s_2}^{2k} = \left\{ d \in I_{2s_1+s_2}^{2k} \mid s_1 + s_2 + H_e(d^+) + H_{\mathbb{Z}_2}(d^+) \le k-1 \ and \ s_1 + s_2 + H_e(d^-) + H_{\mathbb{Z}_2}(d^-) \le k-1 \right\},$$

where

   (i) $s_1 = \natural \left\{ C : C \text{ is a through class of } R^d \text{ such that } H_C^d = \{e\} \right\},$

   (ii) $s_2 = \natural \left\{ C : C \text{ is a through class of } R^d \text{ such that } H_C^d = \mathbb{Z}_2 \right\},$

   (iii) $H_e(d^+) \, (H_e(d^-))$ is the number of $\{e\}$ horizontal edges $C$ in the top(bottom) row of $R^d$ such that $H_C^d = \{e\}$ and $|C| \ge 2$,

   (iv) $H_{\mathbb{Z}_2}(d^+) \, (H_{\mathbb{Z}_2}(d^-))$ is the number of $\mathbb{Z}_2$ horizontal edges $C$ in the top(bottom) row of $R^d$ such that $H_C^d = \mathbb{Z}_2$.

   (v) $\sharp^p(R^d) = s_1 + s_2$.

Put, $\widetilde{I}_r^{2k} = \bigcup\limits_{2s_1+s_2 \le r} \widetilde{I}_{2s_1+s_2}^{2k}$.

**Definition 2.9.** ([5], *Definition 3.1.1*)

When $s_1 = k, r = 2k, \widetilde{I}_r^{2k} = I_{2k}^{2k}$.

Let $\widetilde{I}_{2k} = \bigcup\limits_{r=0}^{2k} \widetilde{I}_r^{2k}$. The linear span of $\widetilde{I}_{2k}$ is denoted by $\mathscr{H}$.

**Theorem 2.10.** ([5], *Theorem 3.1.4 and Theorem 3.1.5*)

   (1) $\mathscr{H}$ is a finite-dimensional subalgebra of $A_k^{\mathbb{Z}_2}(x)$ where $\mathscr{H}$ is as in Definition 2.9.

   (2) The signed partition algebra $\overrightarrow{A}_k^{\mathbb{Z}_2}(x)$ and $\mathscr{H}$ are equal.

**Theorem 2.11.** ([5], *Theorem 3.1.7*)

(i) *The dimension of $A_k^{\mathbb{Z}_2}(x)$ is*

$$\sum n_\lambda \prod_{i=1}^{t} \left(2^{\lambda_i - 1} + 1\right)$$

*where the sum is over the partition $\lambda = (\lambda_1, \cdots, \lambda_t) \vdash 2k$ and $n_\lambda$ is the number of diagrams $d \in \mathbb{P}_k(x)$ such that $\|d\| = \lambda = (\lambda_1, \cdots, \lambda_t)$ be the partition of $2k$, corresponding to the set partition $d$, where $\lambda_i$ is the cardinality of the equivalence class.*

(ii) *The dimension of the signed partition algebra $\overrightarrow{A}_k^{\mathbb{Z}_2}(x)$ is*

$$k! \, 2^k + \sum [(2^r - 1)/2^r]^s \prod_{i \geq 1} (2^{\lambda_i - 1} + 1)$$

*where the sum is over the partition diagrams $d$ in $\mathbb{P}_k(x), \|d\| = \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_t) \to 2k$, $r = k - \sharp^p(d), s = 0$ if $|d^+| \neq k$ and $|d^-| \neq k$, $s = 1$ if and only if $|d^+| = k$ or $|d^-| = k$ and $s = 2$ if $d \notin S_k, |d^+| = |d^-| = k$.*

**Example 2.12.**     (i) *For $k = 1, 2, \cdots$, Dimensions of $A_k^{\mathbb{Z}_2}(x)$ are $7, 164, \cdots$*

(ii) *For $k = 1, 2, 3, \cdots$, Dimensions of $\overrightarrow{A}_k^{\mathbb{Z}_2}(x)$ are $3, 85, 5055, \cdots$*

**Lemma 2.13.** *Let $I_{2k}^{2k}$ be as in Definition 2.9 then $I_{2k}^{2k} \simeq \mathbb{Z}_2 \wr \mathfrak{S}_k$.*

*Proof.* Let $d \in I_{2k}^{2k}$, then $\sharp(d) = 2k$ and $\sharp(R^d) = k$ and $R^d$ is a permutation in $\mathfrak{S}_k$.

Define,

$$f(i) = \begin{cases} \overline{1}, & \text{if } (i, e) \sim (i', g); \\ \overline{0}, & \text{if } (i, e) \sim (i', e). \end{cases}$$

Thus, $d = (f, R^d) \in \mathbb{Z}_2 \wr \mathfrak{S}_k$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

**Theorem 2.14.** *([7], Theorem 3.26)*

Let $R$ be a commutative ring with unity. Let $\Lambda_{s_1} = \{(\lambda_1, \lambda_2) \mid \lambda_1 \vdash k_1, \lambda_2 \vdash k_2, k_1 + k_2 = s_1\}$ and $\Lambda_{s_2} = \{\mu \mid \mu \vdash s_2\}$. For $(\lambda_1, \lambda_2) \in \Lambda_{s_1}$, and $\mu \in \Lambda_{s_2}$ define $M^{(\lambda_1, \lambda_2)}$ and $M^\mu$ be the set of all standard tableaux of shape $(\lambda_1, \lambda_2)$ and $\mu$ respectively.

(i) *The algebra $\mathscr{H} = R[\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}]$ is a free $R$-module with basis*

$\mathscr{M} = \big\{ m_{s_\lambda, t_\lambda}^\lambda \mid s_\lambda$ *and* $t_\lambda$ *are standard tableaux of shape* $\lambda$ *for some bi-partition* $\lambda$ *of* $k$
$\qquad$ *in* $M^{(\lambda_1, \lambda_2)}$ *and* $\lambda = (\lambda_1, \lambda_2)\big\}$

*where $m_{s_\lambda, t_\lambda}^\lambda$ is as in Definition 3.14 of [7].*

*Moreover, $\mathscr{M}$ is a cellular basis for $\mathscr{H}$.*

(ii) *The algebra $\mathscr{H}' = R[\mathfrak{S}_{s_2}]$ is a free $R$-module with basis*

$\mathscr{M}' = \big\{ m_{s_\mu, t_\mu}^\mu \mid s_\mu$ *and* $t_\mu$ *are standard tableaux of shape* $\mu$ *for some partition* $\mu$ *of* $k$ *in* $M^\mu \big\}$
*where $m_{s_\mu, t_\mu}^\mu$ is as in Definition 3.14 of [7].*

*Moreover, $\mathscr{M}$ is a cellular basis for $\mathscr{H}'$.*

*Also, $\mathscr{M}$ is a cellular basis for $\mathscr{H}' \otimes K(x)$, where $K$ is a field.*

**Theorem 2.15.** *Let $R\left[(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}\right]$ be the $R$-algebra, then by Theorem 2.14, $R[(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}] \simeq R[\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}] \otimes R[\mathfrak{S}_{s_2}]$ is a cellular algebra with a cell datum $(\Lambda_{s_1, s_2}, M^{((\lambda_1, \lambda_2), \mu)}, C^{((\lambda_1, \lambda_2), \mu)}, *)$ given as follows:*

(i) $\Lambda_{s_1,s_2} := \{((\lambda_1,\lambda_2),\mu) \mid |\lambda_1|+|\lambda_2|=s_1, \mu \vdash s_2\} \cup \{((\lambda_1,\lambda_2),\Phi) \mid |\lambda_1|+|\lambda_2|=s_1\} \cup \{((\Phi,\Phi),\mu) \mid \mu \vdash s_2\} \cup \{\Phi\}$ *(ordered lexicographically) is a partially ordered set.*

(ii) $M^{((\lambda_1,\lambda_2),\mu)} := \{((s_{\lambda_1},s_{\lambda_2}),s_\mu) \mid s_{\lambda_1}, s_{\lambda_2}$ *and* $s_\mu$ *are the standard tableaus of shape* $\lambda_1,\lambda_2$

*and* $\mu$ *respectively} such that*

$$C^{((\lambda_1,\lambda_2),\mu)} : \coprod_{\lambda,\mu \in \Lambda} M^{((\lambda_1,\lambda_2),\mu)} \times M^{((\lambda_1,\lambda_2),\mu)} \to (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$$

*is an injective map with image an R basis of* $(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$.

(iii) *If* $\lambda = (\lambda_1,\lambda_2)$ *and* $S = ((s_{\lambda_1}, s_{\lambda_2}), s_\mu), T = ((t_{\lambda_1}, t_{\lambda_2}), t_\mu) \in M^{((\lambda_1,\lambda_2),\mu)}$, *write*

$$C(S,T) = m^{\lambda}_{s_\lambda t_\lambda} m^{\mu}_{s_\mu t_\mu}$$

*where* $m^{\lambda}_{s_\lambda t_\lambda}$ *and* $m^{\mu}_{s_\mu t_\mu}$ *are as in Theorem 2.14.* $*$ *is the anti-automorphism of* $(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ *such that* $((f,\sigma_1),\sigma_2)^* = ((f,\sigma_1)^*, \sigma_2^*) = ((f,\sigma_1)^{-1}, \sigma_2^{-1}) \; \forall \; ((f,\sigma_1),\sigma_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ *such that* $\left(C^{((\lambda_1,\lambda_2),\mu)}(S,T)\right)^* = C^{((\lambda_1,\lambda_2),\mu)}(T,S)$.

## 3. Differences between the algebras

In this section, we illustrate that the algebra of $\mathbb{Z}_2$-relations $A_k^{\mathbb{Z}_2}(x)$ and signed partition algebras $\overrightarrow{A}_k^{\mathbb{Z}_2}(x)$ are different from the $\mathbb{Z}_2$-colored partition algebra $P_k(x;\mathbb{Z}_2)$ introduced in [4] and Tanabe algebras $T_{k,m}(x)$ introduced in [8].

**Example 3.1.** *This example clearly illustrates that the signed partition algebras are different from* $\mathbb{Z}_2$*-colored partition algebra introduced in* [4].

| | | |
|---|---|---|
| fig 47, fig 48, fig 49 | | fig 37′, fig 38′ |
| fig 50, fig 51, fig 52 | | fig 39′, fig 40′ |
| fig 53, fig 54, fig 55 | | fig 41′, fig 42′ |
| fig 56, fig 57, fig 58 | | fig 43′, fig 44′ |
| fig 59, fig 60, fig 61, fig 62, fig 63, fig 64, fig 65, fig 66, fig 67 | | fig 45′, fig 46′ |
| fig 68, fig 69, fig 70, fig 71, fig 72, fig 73, fig 74, fig 75, fig 76 | | fig 47′, fig 48′ |
| fig 77, fig 78, fig 79, fig 80, fig 81, fig 82, fig 83, fig 84, fig 85 | | fig 49′ |

**Note 1.** *In the algebra of $\mathbb{Z}_2$-relations and signed partition algebras, the set of all diagrams having no horizontal edges and each through class contains two vertices is isomorphic to the hyperoctahedral group of type $B_n$ whereas in Tanabe algebras, the set of all diagrams having no horizontal edges and each through class contains two vertices is isomorphic to the symmetric group.*

*Thus, the representations of algebra of $\mathbb{Z}_2$ relations and signed partition algebras are determined by the representations of hyperoctahedral group of type $B_n$ whereas the representations of Tanabe algebras are determined by the representations of symmetric group.*

## 4. The algebra of $\mathbb{Z}_2$ relations and signed partition algebras as Tabular algebras

In this section, we realize the algebra of $\mathbb{Z}_2$ relations and signed partition algebras as tabular algebras introduced in [2].

**Notation 4.1.** *Let $d \in I^{2k}_{2s_1+s_2}\left(\widetilde{d} \in \widetilde{I}^{2k}_{2s_1+s_2}\right)$, be as in Definition 2.5(2.7).*

(i) *The vertex having least integer value in a connected component of $d(\widetilde{d})$ is called the minimal vertex of the connected component.*

(ii) *$|d|(|\widetilde{d}|)$ denotes the number of connected components in $d(\widetilde{d})$.*

**Definition 4.2.** *Define,*

(i) *$M[(r,(s_1,s_2))] = \Big\{(d,P) \mid d \in R^{\mathbb{Z}_2}_k, P \in R^{\mathbb{Z}_2}_{k'} \text{ and } d \setminus P \in R^{\mathbb{Z}_2}_{k-k'}, |d| \geq 2s_1+s_2, P \text{ is a subset of the set of all connected components of $d$ with } |P| = 2s_1 + s_2 \text{ where } r = 2s_1 + s_2,$*

$P = P_1^e \cup P_1^g \cup \cdots \cup P_{s_1}^e \cup P_{s_1}^g \cup P_1^{\mathbb{Z}_2} \cup \cdots \cup P_{s_2}^{\mathbb{Z}_2}$ such that $H^d_{R^{P_i^{\{e\}}}} = \{e\}, 1 \le i \le s_1,$

$H^d_{R^{P_j^{\mathbb{Z}_2}}} = \mathbb{Z}_2, 1 \le j \le s_2 \Big\}.$

(ii) $\widetilde{M}[(r,(s_1,s_2))] = \Big\{ (\widetilde{d}, \widetilde{P}) \mid \widetilde{d} \in R_k^{\mathbb{Z}_2}, \widetilde{P} \in R_{k'}^{\mathbb{Z}_2}$ and $\widetilde{d} \setminus \widetilde{P} \in R_{k-k'}^{\mathbb{Z}_2}, |d| \ge 2s_1 + s_2, \widetilde{P}$ is a subset of the

set of all connected components of $\widetilde{d}$ with $|\widetilde{P}| = 2s_1 + s_2$ where $r = 2s_1 + s_2,$

$\widetilde{P} = \widetilde{P}_1^e \cup \widetilde{P}_1^g \cup \cdots \cup \widetilde{P}_{s_1}^e \cup \widetilde{P}_{s_1}^g \cup \widetilde{P}_1^{\mathbb{Z}_2} \cup \cdots \cup \widetilde{P}_{s_2}^{\mathbb{Z}_2}$ such that $H^d_{R^{\widetilde{P}_i^{\{e\}}}} = \{e\}, 1 \le i \le s_1,$

$H^d_{R^{\widetilde{P}_j^{\mathbb{Z}_2}}} = \mathbb{Z}_2, 1 \le j \le s_2$ and $2r_1(r_2)$ is the number of $\{e\}(\mathbb{Z}_2)$ connected components in $\widetilde{d} \setminus \widetilde{P},$

$s_1 + s_2 + r_1 + r_2 \le k - 1$ if $s_1 + s_2 + r_1 + r_2 = k$ then $s_1 = k$ or $r_1 \ne 0 \Big\}.$

We shall now introduce an ordering for the connected components in $P$.

Suppose that $P = P_1^e \cup P_1^g \cup \cdots \cup P_{s_1}^e \cup P_{s_1}^g \cup P_1^{\mathbb{Z}_2} \cup \cdots \cup P_{s_2}^{\mathbb{Z}_2}$ then $R^P = R^{P_1^{\{e\}}} \cup \cdots \cup R^{P_{s_1}^{\{e\}}} \cup R^{P_1^{\mathbb{Z}_2}} \cup \cdots \cup R^{P_{s_2}^{\mathbb{Z}_2}}.$

Let $a_{11}, \cdots, a_{1s_1}$ be the minimal vertices of the connected components $R^{P_1^{\{e\}}}, \cdots, R^{P_{s_1}^{\{e\}}}$ in $R^P$ and

$b_{11}, \cdots, b_{1s_2}$ be the minimal vertices of the connected components $R^{P_1^{\mathbb{Z}_2}}, \cdots, R^{P_{s_2}^{\mathbb{Z}_2}}$ in $R^P$ then

$P_i^e < P_j^e$ and $P_i^g < P_j^g$ if and only if $R^{P_i^{\{e\}}} < R^{P_j^{\{e\}}}$ if and only if $a_{1i} < a_{1j} \in R^P$ and

$P_l^{\mathbb{Z}_2} < P_f^{\mathbb{Z}_2}$ if and only if $R^{P_l^{\mathbb{Z}_2}} < R^{P_f^{\mathbb{Z}_2}}$ if and only if $b_{1l} < b_{1f} \in R^P.$

Similarly, we can introduce an ordering for the connected components in $\widetilde{P}$ as in $P$.

**Lemma 4.3.** *Let $M[(r,(s_1,s_2))]$ and $\widetilde{M}[(r,(s_1,s_2))]$ be as in Definition 4.2.*

(i) *Each $d \in I_{2s_1+s_2}^{2k}$ can be associated with a pair of elements $(d^+, P), (d^-, Q) \in M[(r,(s_1,s_2))]$ and an element $((f,\sigma_1),\sigma_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ where $(d^+, P), (d^-, Q) \in M[(r,(s_1,s_2))]$ and $((f,\sigma_1),\sigma_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}.$*

(ii) *Each $\widetilde{d} \in \widetilde{I}_{2s_1+s_2}^{2k}$ can be associated with a pair of elements $(\widetilde{d}^+, \widetilde{P}), (\widetilde{d}^-, \widetilde{Q}) \in \widetilde{M}[(r,(s_1,s_2))]$ and an element $((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ where $(\widetilde{d}^+, \widetilde{P}), (\widetilde{d}^-, \widetilde{Q}) \in \widetilde{M}[(r,(s_1,s_2))]$ and $((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}.$*

*Proof.* **Proof of (i):** Let $d \in I_{2s_1+s_2}^{2k}.$

$d^+, d^-$ are the diagrams obtained from the diagram $d$ by restricting the vertex set to

$\{(1,e),(1,g),\cdots,(k,e),(k,g)\}$ and $\{(1',e),(1',g),\cdots,(k',e),(k',g)\}$ respectively.

Identifying $\{(1',e),(1',g),\cdots,(k',e),(k',g)\}$ with $\{(1,e),(1,g),\cdots,(k,e),(k,g)\}$ by sending

$$(i',e) \mapsto (i,e) \text{ and } (i',g) \mapsto (i,g).$$

Thus, $d^+, d^- \in R_k^{\mathbb{Z}_2}.$

Let $S_d$ be the set of all through classes of $d$. Let $P$ denote the set of all connected components obtained from $S_d$ by restricting the vertex set to $\{(1,e),(1,g),\cdots,(k,e),(k,g)\}$. i.e., $S_d \cap d^+ = P.$

Thus, $|P| = 2s_1 + s_2.$

Similarly, let $Q$ denote the set of all connected components obtained from $S_d$ by restricting the vertex set to $\{(1',e),(1',g),\cdots,(k',e),(k',g)\}$. i.e., $S_d \cap d^- = Q.$

Identify $\{(1',e),(1',g),\cdots,(k',e),(k',g)\}$ with $\{(1,e),(1,g),\cdots,(k,e),(k,g)\}$ by sending

$$(i',e) \mapsto (i,e) \text{ and } (i',g) \mapsto (i,g).$$

Thus, $|Q| = 2s_1 + s_2$.

Write

$$P = P_1^e \cup P_1^g \cup \cdots P_{s_1}^e \cup P_{s_1}^g \cup P_1^{\mathbb{Z}_2} \cup \cdots P_{s_2}^{\mathbb{Z}_2} \text{ and}$$

$$Q = Q_1^e \cup Q_1^g \cup \cdots \cup Q_{s_1}^e \cup Q_{s_1}^g \cup Q_1^{\mathbb{Z}_2} \cup \cdots Q_{s_2}^{\mathbb{Z}_2}$$

Define an element $(f, \sigma_1)$ as follows:

If there is a connected component $X \in S_d$ containing $P_i^e$ and $Q_j^{g'}, g' \in \mathbb{Z}_2$

then, define $\sigma_1(i) = j$ and

$$f(i) = \begin{cases} \overline{1}, & \text{if } g' = g; \\ \overline{0}, & \text{if } g' = e. \end{cases}$$

Thus, $(f, \sigma_1) \in \mathbb{Z}_2 \wr \mathfrak{S}_{s_1}$.

Similarly, define $\sigma_2$ as follows:

If there is a connected component $Y \in S_d$ containing $P_l^{\mathbb{Z}_2}$ and $Q_m^{\mathbb{Z}_2}$ then, define $\sigma_2(l) = m$.

Thus, $\sigma_2 \in \mathfrak{S}_{s_2}$ which implies that $((f, \sigma_1), \sigma_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$.

**Proof of (ii):** By Definition 2.8, $s_1 + s_2 + H_e(\widetilde{d}^+) + H_{\mathbb{Z}_2}(\widetilde{d}^+) \leq k - 1$ and $s_1 + s_2 + H_e(\widetilde{d}^-) + H_{\mathbb{Z}_2}(\widetilde{d}^-) \leq k - 1$ and the proof of (ii) is same as proof of (i). $\square$

**Lemma 4.4.**     (i) *For every pair* $(d', P), (d'', Q) \in M[(r, (s_1, s_2))]$ *and an element*
$((f, \sigma_1), \sigma_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ *there is a unique diagram* $d \in I_{2s_1+s_2}^{2k}$ *where* $d^+ = (d', P), d^- = (d'', Q)$
*such that there is a unique connected component of $d$ containing $P_i^e$ and $Q_{\sigma_1(i)}^{g'}$ and $P_j^{\mathbb{Z}_2}$ and $Q_{\sigma_2(j)}^{\mathbb{Z}_2}$.*

  (ii) *For every pair* $(\widetilde{d}', \widetilde{P}), (\widetilde{d}'', \widetilde{Q}) \in \widetilde{M}[(r, (s_1, s_2))]$ *and an element* $((\widetilde{f}, \widetilde{\sigma}_1), \widetilde{\sigma}_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ *there
is a unique diagram* $\widetilde{d} \in \widetilde{I}_{2s_1+s_2}^{2k}$ *where* $\widetilde{d}^+ = (\widetilde{d}', \widetilde{P}), \widetilde{d}^- = (\widetilde{d}'', \widetilde{Q})$ *such that there is a unique
connected component of $\widetilde{d}$ containing $\widetilde{P}_i^e$ and $\widetilde{Q}_{\sigma_1(i)}^{g'}$ and $\widetilde{P}_j^{\mathbb{Z}_2}$ and $\widetilde{Q}_{\sigma_2(j)}^{\mathbb{Z}_2}$.*

*Proof.* **Proof of (i):** Let $P = P_1^e \cup P_1^g \cup \cdots \cup P_{s_1}^e \cup P_{s_1}^g \cup P_1^{\mathbb{Z}_2} \cup \cdots P_{s_2}^{\mathbb{Z}_2}$ and $Q = Q_1^e \cup Q_1^g \cup \cdots \cup Q_{s_1}^e \cup Q_{s_1}^g \cup$
$Q_1^{\mathbb{Z}_2} \cup \cdots \cup Q_{s_2}^{\mathbb{Z}_2}$.

Let $\{a_{11}^e, \cdots, a_{1s_1}^e\}, \{a_{11}^g, \cdots, a_{1s_1}^g\}$ and $\{b_{11}^e, \cdots, b_{1s_2}^e\}$ be the minimal vertices of the connected compo-
nents $\{R^{P_1^e}, \cdots, R^{P_{s_1}^e}\}, \{R^{P_1^g}, \cdots, R^{P_{s_1}^g}\}$ and $\{R^{P_1^{\mathbb{Z}_2}}, \cdots, R^{P_{s_2}^{\mathbb{Z}_2}}\}$ respectively.

Similarly, let $\{l_{11}^e, \cdots, l_{1s_1}^e\}, \{l_{11}^g, \cdots, l_{1s_1}^g\}$ and $\{f_{11}^e, \cdots, f_{1s_2}^e\}$ be the minimal vertices of the connected
components $\{R^{Q_1^e}, \cdots, R^{Q_{s_1}^e}\}, \{R^{Q_1^g}, \cdots, R^{Q_{s_1}^g}\}$ and $\{R^{Q_1^{\mathbb{Z}_2}}, \cdots, R^{Q_{s_2}^{\mathbb{Z}_2}}\}$ respectively.

Let $d \in \widetilde{I}_{2s_1+s_2}^{2k}$ be obtained as follows:

  (i) Draw $(d', P)$ above $(d'', Q)$.

 (ii) Connect $P_i^e$ to $Q_{\sigma_1(i)}^{g'}$ if $f(i) = g'$. Also, connect $P_j^{\mathbb{Z}_2}$ to $Q_{\sigma_2(j)}^{\mathbb{Z}_2}$.

(iii) All other connected components in $(d', P)$ $((d'', Q))$ other than the connected components of $P(Q)$
     will remain as horizontal edges or isolated points in the top(bottom) row of $d \in \widetilde{I}_{2s_1+s_2}^{2k}$, by our
     construction $d^+ = (d', P)$ and $d^- = (d'', Q)$.

**Proof of (ii):** Proof of (ii) is similar to the proof of (i). $\square$

**Remark 4.5.** *By Lemma 4.3, any* $d \in I_{2k} (\widetilde{d} \in \widetilde{I}_{2k})$, *is denoted by* $C_{(d^+,P),(d^-,Q)}^{((f,\sigma_1),\sigma_2)} \left( \widetilde{C}_{(\widetilde{d}^+,\widetilde{P}),(\widetilde{d}^-,\widetilde{Q})}^{((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)} \right)$.

**Definition 4.6.** (i) *Define a map* $\phi^r_{s_1,s_2} : M[(r,(s_1,s_2))] \times M[(r,(s_1,s_2))] \to R[(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}]$ *as follows:*

$$\phi^r_{s_1,s_2}((d',P),(d'',Q)) = x^{l(P \vee Q)}((f,\sigma_1),\sigma_2) \text{ and}$$

(ii) *Define a map* $\widetilde{\phi}^r_{s_1,s_2} : \widetilde{M}[(r,(s_1,s_2))] \times \widetilde{M}[(r,(s_1,s_2))] \to R[(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}]$ *as follows:*

$$\widetilde{\phi}^r_{s_1,s_2}\left((\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q})\right) = x^{l(\widetilde{P} \vee \widetilde{Q})}((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)$$

*if*

(a) *No two connected components of $Q(\widetilde{Q})$ in $d''(\widetilde{d}'')$ have non-empty intersection with a common connected component of $d'(\widetilde{d}')$ in $d'.d''(\widetilde{d}'.\widetilde{d}'')$, or vice versa.*

(b) *No connected component of $Q(\widetilde{Q})$ has non-empty intersection only with the connected components excluding the connected components of $P(\widetilde{P})$ in $d'.d''(\widetilde{d}'.\widetilde{d}'')$. Similarly, no connected component in $P(\widetilde{P})$ has non-empty intersection only with a connected component excluding the connected components of $Q(\widetilde{Q})$ in $d'.d''(\widetilde{d}'.\widetilde{d}'')$.*

*where $l(P \vee Q) \left( l(\widetilde{P} \vee \widetilde{Q}) \right)$ denotes the number of connected components in $d'.d''(\widetilde{d}'.\widetilde{d}'')$ excluding the union of all the connected components of $P(\widetilde{P})$ and $Q(\widetilde{Q})$.*

*The permutation $((f,\sigma_1),\sigma_2) \left( ((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2) \right)$ is obtained as follows: If there is a unique connected component in $d'.d''(\widetilde{d}'.\widetilde{d}'')$ containing $P_i^e(\widetilde{P}_i^e)$ and $Q_j^{g'}(\widetilde{Q}_j^{g'})$ then, define $\sigma_1(i) = j(\widetilde{\sigma}_1(i) = j)$ and*

$$f(i) = \widetilde{f}(i) = \begin{cases} \overline{1}, & \text{if } g' = g; \\ \overline{0}, & \text{if } g' = e. \end{cases}$$

*Also, if there is a unique connected component in $d'.d''(\widetilde{d}'.\widetilde{d}'')$ containing $P_l^{\mathbb{Z}_2}$ and $Q_f^{\mathbb{Z}_2}(\widetilde{P}_l^{\mathbb{Z}_2}$ and $\widetilde{Q}_f^{\mathbb{Z}_2})$ then, define $\sigma_2(l) = f(\widetilde{\sigma}_2(l) = f)$.*

*Otherwise, $\phi^r_{s_1,s_2}((d',P),(d'',Q)) = 0 \left( \widetilde{\phi}^r_{s_1,s_2}\left((\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q})\right) = 0 \right)$.*

Since the algebra of $\mathbb{Z}_2$-relations and signed partition algebras are subalgebras of partition algebras the proof of Lemmas 4.7 and 4.8 follow as in [10].

**Lemma 4.7.** (i) *Let $\mu, \nu \in I^{2k}_{2s_1+s_2}$ then $\sharp^p(\mu\nu) \le 2s_1 + s_2$. If $\sharp^p(\mu\nu) = 2s_1 + s_2$ then*

$$\mu\nu = C^{r_\mu[(d,R),(d'',Q)]((f',\sigma_1'),\sigma_2')}_{((d,R),(d''',T))}$$

*where $\mu = C^{((f,\sigma_1),\sigma_2)}_{(d,R),(d',P)}, \nu = C^{((f',\sigma_1'),\sigma_2')}_{(d'',Q),(d''',T)}, (d,R),(d',P),(d'',Q),(d''',T) \in M[(r,(s_1,s_2))], ((f,\sigma_1),\sigma_2),$ $((f',\sigma_1'),\sigma_2') \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}, r_\mu[(d,R),(d'',Q)] = ((f,\sigma_1),\sigma_2)\phi^r_{s_1,s_2}[(d',P),(d'',Q)]$ and $r_\mu[(d,R),(d'',Q)]$ is independent of $(d''',T)$ and $((f',\sigma_1'),\sigma_2')$.*

(ii) *Let $\widetilde{\mu}, \widetilde{\nu} \in \widetilde{I}^{2k}_{2s_1+s_2}$ then $\sharp^p(\widetilde{\mu}\widetilde{\nu}) \le 2s_1 + s_2$. If $\sharp^p(\widetilde{\mu}\widetilde{\nu}) = 2s_1 + s_2$ then*

$$\widetilde{\mu}\widetilde{\nu} = C^{r_{\widetilde{\mu}}[(\widetilde{d},\widetilde{R}),(\widetilde{d}'',\widetilde{Q})]((\widetilde{f}',\widetilde{\sigma}_1'),\widetilde{\sigma}_2')}_{((\widetilde{d},\widetilde{R}),(\widetilde{d}''',\widetilde{T}))}$$

*where $\widetilde{\mu} = C^{((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)}_{(\widetilde{d},\widetilde{R}),(\widetilde{d}',\widetilde{P})}, \widetilde{\nu} = \widetilde{C}^{((\widetilde{f}',\widetilde{\sigma}_1'),\widetilde{\sigma}_2')}_{(\widetilde{d}'',\widetilde{Q}),(\widetilde{d}''',\widetilde{T})}, (\widetilde{d},\widetilde{R}),(\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q}),(\widetilde{d}''',\widetilde{T}) \in \widetilde{M}[(r,(s_1,s_2))], ((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2),$ $((\widetilde{f}',\widetilde{\sigma}_1'),\widetilde{\sigma}_2') \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}, r_{\widetilde{\mu}}[(\widetilde{d},\widetilde{R}),(\widetilde{d}'',\widetilde{Q})] = ((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)\widetilde{\phi}^r_{s_1,s_2}[(\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q})]$ and $r_{\widetilde{\mu}}[(\widetilde{d},\widetilde{R}),(\widetilde{d}'',\widetilde{Q})]$ is independent of $(\widetilde{d}''',\widetilde{T})$ and $((\widetilde{f}',\widetilde{\sigma}_1'),\widetilde{\sigma}_2')$.*

*Proof.* If $\sharp^p(\mu\nu) = 2s_1 + s_2$, then the proof of (i) and (ii) follows from the definition of multiplication of partition algebras and Definition 4.6 and from Lemma 4.4 of [10].

$\square$

**Lemma 4.8.** (i) *Let* $\mu \in I^{2k}_{2s'_1+s'_2}, \nu \in I^{2k}_{2s_1+s_2}$ *then* $\sharp^p(\mu\nu) \leq 2s_1 + s_2$. *If* $\sharp^p(\mu\nu) = 2s_1 + s_2$ *then*

$$\mu\nu = x^{l(P\vee Q)}C^{r_\mu[(w,F),(d'',Q)]((f',\sigma'_1),\sigma'_2)}_{(w,F),(d''',T)}$$

*where* $\mu = C^{((f,\sigma_1),\sigma_2)}_{(d,R),(d',P)}, \nu = C^{((f',\sigma'_1),\sigma'_2)}_{(d'',Q),(d''',T)}, (d,R),(d',P) \in M[(r',(s'_1,s'_2))],(w,F),(d'',Q),$
$(d''',T) \in M[(r,(s_1,s_2))], ((f,\sigma_1),\sigma_2) \in \left(\mathbb{Z}_2 \wr \mathfrak{S}_{s'_1}\right) \times \mathfrak{S}_{s'_2}, ((f',\sigma'_1),\sigma'_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}, r_\mu[(w,F),(d'',Q)]$
*is independent of* $((f',\sigma'_1),\sigma'_2)$ *and* $(d''',T)$.

(ii) *Let* $\widetilde{\mu} \in \widetilde{I}^{2k}_{2s'_1+s'_2}, \widetilde{\nu} \in \widetilde{I}^{2k}_{2s_1+s_2}$ *then* $\sharp^p(\widetilde{\mu}\widetilde{\nu}) \leq 2s_1 + s_2$. *If* $\sharp^p(\widetilde{\mu}\widetilde{\nu}) = 2s_1 + s_2$ *then*

$$\widetilde{\mu}\widetilde{\nu} = x^{l(\widetilde{P}\vee\widetilde{Q})}C^{r_{\widetilde{\mu}}[(\widetilde{w},\widetilde{F}),(\widetilde{d}'',\widetilde{Q})]((\widetilde{f}',\widetilde{\sigma}'_1),\widetilde{\sigma}'_2)}_{(\widetilde{w},\widetilde{F}),(\widetilde{d}''',\widetilde{T})}$$

*where* $\widetilde{\mu} = \widetilde{C}^{((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)}_{(\widetilde{d},\widetilde{R}),(\widetilde{d}',\widetilde{P})}, \widetilde{\nu} = \widetilde{C}^{((\widetilde{f}',\widetilde{\sigma}'_1),\widetilde{\sigma}'_2)}_{(\widetilde{d}'',\widetilde{Q}),(\widetilde{d}''',\widetilde{T})}, (\widetilde{d},\widetilde{R}),(\widetilde{d}',\widetilde{P}) \in \widetilde{M}[(r',(s'_1,s'_2))],(\widetilde{w},\ widetilde F),(\widetilde{d}'',\widetilde{Q}),(\widetilde{d}''',\widetilde{T}) \in$
$\widetilde{M}[(r,(s_1,s_2))], ((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2) \in \left(\mathbb{Z}_2 \wr \mathfrak{S}_{s'_1}\right) \times \mathfrak{S}_{s'_2}, ((\widetilde{f}',\widetilde{\sigma}'_1),\widetilde{\sigma}'_2) \in (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}, r_{\widetilde{\mu}}[(\widetilde{w},\widetilde{F}),(\widetilde{d}'',\widetilde{Q})]$
*is independent of* $((\widetilde{f}',\widetilde{\sigma}'_1),\widetilde{\sigma}'_2)$ *and* $(\widetilde{d}''',\widetilde{T})$.

*Proof.* **Proof of (i):**If $\sharp^p\left(C^{((f,\sigma_1),\sigma_2)}_{(d,R),(d',P)} C^{((f',\sigma'_1),\sigma'_2)}_{(d'',Q),(d''',T)}\right) = 2s_1 + s_2$, then by Lemma 4.4 and [10] there exists
$(w,F),(v,Q) \in M[(r,(s_1,s_2))], ((f'',\sigma''_1),\sigma''_2) \in \mathbb{Z}_2 \wr \mathfrak{S}_{s_1} \times \mathfrak{S}_{s_2}.$
$$C^{((f,\sigma_1),\sigma_2)}_{(d,R),(d',P)} C^{((f',\sigma'_1),\sigma'_2)}_{(d'',Q),(d''',T)} = C^{r_\mu[(w,F),(d'',Q)]((f',\sigma'_1),\sigma'_2)}_{(w,F),(v,Q)}$$
where $\mu = C^{((f,\sigma_1),\sigma_2)}_{(d,R),(d',P)} r_\mu[(w,F),(d'',Q)]((f',\sigma'_1),\sigma'_2) = x^{l(P\vee Q)}((f'',\sigma''_1),\sigma''_2)$ and it is independent of $((f',\sigma'_1),\sigma'_2)$ and $(d''',T)$.

**Proof of (ii):** Proof of (ii) is same as that of proof of (i). $\square$

**Definition 4.9.** *Put,*

(i) $\Lambda = \{(r,(s_1,s_2)) \mid r = 2s_1 + s_2, 0 \leq s_1, s_2 \leq k\}$ *and*

(ii) $\widetilde{\Lambda} = \{(r,(s_1,s_2)) \mid r = 2s_1 + s_2, 0 \leq s_1 \leq k, 0 \leq s_2 \leq k-1\}$.

*Define a relation '$\leq$' on* $\Lambda(\widetilde{\Lambda})$ *as follows:*

$$(r,(s_1,s_2)) \leq (r',(s'_1,s'_2))$$

*if and only if*

(a) $r < r'$ *or*

(b) $r' = r$ *and* $s_1 + s_2 < s'_1 + s'_2$

*Thus* $(\Lambda,\leq)((\widetilde{\Lambda},\leq))$ *is a partially ordered set.*

**Note 2.** *Let* $B^r(s_1,s_2) = B(s_1,s_2) = (\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}$ *and* $\Gamma_{s_1,s_2} = \mathscr{A}[(\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}) \times \mathfrak{S}_{s_2}]$, *where* $r = 2s_1 + s_2$. *The elements of* $B(s_1,s_2)$ *forms a basis of* $\Gamma(s_1,s_2)$. *Thus* $(\Gamma(s_1,s_2), B(s_1,s_2))$ *is a hyper group.*

**Definition 4.10.** *Let* $M[(r,(s_1,s_2))] \left(\widetilde{M}[(r,(s_1,s_2))]\right)$ *be as in Definition 4.2.*

*Define maps,*

(i) $C : M[(r, (s_1, s_2))] \times B(s_1, s_2) \times M[(r, (s_1, s_2))] \to A_k^{\mathbb{Z}_2}$ as follows:

$$C[(d', P), ((f, \sigma_1), \sigma_2), (d'', Q)] = d,$$

where $d = C_{(d',P),(d'',Q)}^{((f,\sigma_1),\sigma_2)}$, as in Remark 4.5 and $d \in I_{2s_1+s_2}^{2k}$.

By Lemma 4.4, it is clear that $C$ is injective.

(ii) $\widetilde{C} : \widetilde{M}[(r, (s_1, s_2))] \times B(s_1, s_2) \times \widetilde{M}[(r, (s_1, s_2))] \to \overrightarrow{A}_k^{\mathbb{Z}_2}$, as follows:

$$\widetilde{C}\left[(\widetilde{d}', \widetilde{P}), ((\widetilde{f}, \widetilde{\sigma}_1), \widetilde{\sigma}_2), (\widetilde{d}'', \widetilde{Q})\right] = \widetilde{d},$$

where $\widetilde{d} = \widetilde{C}_{(\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q})}^{((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)}$, as in Remark 4.5 and $\widetilde{d} \in \widetilde{I}_{2s_1+s_2}^{2k}$.

By Lemma 4.4, it is clear that $\widetilde{C}$ is injective.

**Definition 4.11.** *Define,*

(i) $* : A_k^{\mathbb{Z}_2} \to A_k^{\mathbb{Z}_2}$ as follows:

$$\left(C_{(d',P),(d'',Q)}^{((f,\sigma_1),\sigma_2)}\right)^* = \left(C_{(d',P),(d'',Q)}^{((f,\sigma_1),\sigma_2)}\right)^f = C_{(d'',Q),(d',P)}^{((f,\sigma_1),\sigma_2)^{-1}}$$

where $f$ is the flip of the diagram and inverse mapping is the anti-automorphism of the hyper group $(\Gamma(s_1, s_2), B(s_1, s_2))$.

Clearly, $*$ is an involutary anti-automorphism of $A_k^{\mathbb{Z}_2}$.

(ii) $\widetilde{*} : \overrightarrow{A}_k^{\mathbb{Z}_2} \to \overrightarrow{A}_k^{\mathbb{Z}_2}$ as follows:

$$\left(\widetilde{C}_{(\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q})}^{((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)}\right)^{\widetilde{*}} = \left(\widetilde{C}_{(\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q})}^{((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)}\right)^f = \widetilde{C}_{(\widetilde{d}'',\widetilde{Q}),(\widetilde{d}',\widetilde{P})}^{((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)^{-1}}$$

where $f$ is the flip of the diagram and inverse mapping is the anti-automorphism of the hyper group $(\Gamma(s_1, s_2), B(s_1, s_2))$.

Clearly, $*$ is an involutary anti-automorphism of $\overrightarrow{A}_k^{\mathbb{Z}_2}$.

**Notation 4.12.** *If $b \in \Gamma(s_1, s_2)$ such that $b = \sum\limits_{((f_i,\sigma_{1_i}),\sigma_{2_i}) \in B(s_1,s_2)} c_i((f_i, \sigma_{1_i}), \sigma_{2_i})$ for some scalars.*

(i) *We write $C_{(d',P),(d'',Q)}^b \in \mathscr{A}\left[A_k^{\mathbb{Z}_2}\right]$ as shorthand for* $\sum\limits_{((f_i,\sigma_{1_i}),\sigma_{2_i}) \in B(s_1,s_2)} c_i C_{(d',P),(d'',Q)}^{((f_i,\sigma_{1_i}),\sigma_{2_i})}$.

*Also, write $C_{s_1,s_2}$ for the image under $C$ of $M[(r, (s_1, s_2))] \times B(s_1, s_2) \times M[(r, (s_1, s_2))]$.*

(ii) *We write $\widetilde{C}_{(\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q})}^b \in \mathscr{A}\left[\overrightarrow{A}_k^{\mathbb{Z}_2}\right]$ as shorthand for* $\sum\limits_{((\widetilde{f}_i,\widetilde{\sigma}_{1_i}),\widetilde{\sigma}_{2_i}) \in B(s_1,s_2)} c_i \widetilde{C}_{(\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q})}^{((\widetilde{f}_i,\widetilde{\sigma}_{1_i}),\widetilde{\sigma}_{2_i})}$.

*Also, write $\widetilde{C}_{s_1,s_2}$ for the image under $\widetilde{C}$ of $\widetilde{M}[(r, (s_1, s_2))] \times B(s_1, s_2) \times \widetilde{M}[(r, (s_1, s_2))]$.*

**Theorem 4.13.** *Let $\mathscr{A} = \mathbb{C}(x)$.*

(i) *An algebra of $\mathbb{Z}_2$-relations $\mathscr{A}[A_k^{\mathbb{Z}_2}]$ is a tabular algebra together with a table datum $(\Lambda, \Gamma, B, M[(r, (s_1, s_2))], C, *)$ where :*

(a) *$\Lambda$ is a finite poset where $\Lambda$ is as in Definition 4.9. For each $(r, (s_1, s_2)) \in \Lambda$, $(\Gamma(s_1, s_2), B(s_1, s_2))$ is a hypergroup over $\mathbb{C}$ and $M[(r, (s_1, s_2))]$ is a finite set. The map*

$$C : \coprod_{(r,(s_1,s_2)) \in \Lambda} (M[(r, (s_1, s_2))] \times B(s_1, s_2) \times M[(r, (s_1, s_2))]) \to A_k^{\mathbb{Z}_2}$$

*is injective with image an $\mathscr{A}$-basis of $A_k^{\mathbb{Z}_2}$.*

(b) *$*$ is an $\mathscr{A}$-linear involutary anti-automorphism of $A_k^{\mathbb{Z}_2}$.*

(c) If $(r, (s_1, s_2)) \in \Lambda, ((f, \sigma_1), \sigma_2) \in \Gamma(s_1, s_2)$ and $(d', P), (d'', Q) \in M[(r, (s_1, s_2))]$ then for all $a \in A_k^{\mathbb{Z}_2}$ we have

$$aC_{(d',P),(d'',Q)}^{((f,\sigma_1),\sigma_2)} \equiv \sum_{(d_i''', R_i) \in M[(r,(s_1,s_2))]} C_{(d_i''',R_i),(d'',Q)}^{r_a[(d_i''',R_i),(d',P)]((f,\sigma_1),\sigma_2)} \quad mod \ A_k^{\mathbb{Z}_2}(< (r, (s_1, s_2))),$$

where $r_a[(d_i''', R_i), (d', P)]((f, \sigma_1), \sigma_2)$ is independent of $(d'', Q)$ and of $((f, \sigma_1), \sigma_2)$.

(ii) An algebra of signed partition algebras $\mathscr{A}[\overrightarrow{A}_k^{\mathbb{Z}_2}]$ is a tabular algebra together with a table datum $(\widetilde{\Lambda}, \Gamma, B, \widetilde{M}[(r, (s_1, s_2))], \widetilde{C}, \widetilde{*})$ where :

(a) $\widetilde{\Lambda}$ is a finite poset where $\widetilde{\Lambda}$ is as in Definition 4.9. For each $(r, (s_1, s_2)) \in \widetilde{\Lambda}$, $(\Gamma(s_1, s_2), B(s_1, s_2))$ is a hypergroup over $\mathbb{C}$ and $\widetilde{M}[(r, (s_1, s_2))]$ is a finite set. The map

$$\widetilde{C} : \coprod_{(r,(s_1,s_2)) \in \widetilde{\Lambda}} (\widetilde{M}[(r, (s_1, s_2))] \times B(s_1, s_2) \times \widetilde{M}[(r, (s_1, s_2))]) \to \overrightarrow{A}_k^{\mathbb{Z}_2}$$

is injective with image an $\mathscr{A}$-basis of $\overrightarrow{A}_k^{\mathbb{Z}_2}$.

(b) $\widetilde{*}$ is an $\mathscr{A}$-linear involutary anti-automorphism of $\overrightarrow{A}_k^{\mathbb{Z}_2}$.

(c) If $(r, (s_1, s_2)) \in \widetilde{\Lambda}, ((f, \sigma_1), \sigma_2) \in \Gamma(s_1, s_2)$ and $\left((\widetilde{d}', \widetilde{P}), (\widetilde{d}'', \widetilde{Q}) \in \widetilde{M}[(r, (s_1, s_2))]\right)$ then for all $\widetilde{a} \in \overrightarrow{A}_k^{\mathbb{Z}_2}$ we have

$$\widetilde{a}\widetilde{C}_{(\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q})}^{((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)} \equiv \sum_{(\widetilde{d}_i''',\widetilde{R}_i) \in \widetilde{M}[(r,(s_1,s_2))]} \widetilde{C}_{(\widetilde{d}_i''',\widetilde{R}_i),(\widetilde{d}'',\widetilde{Q})}^{r_{\widetilde{a}}[(\widetilde{d}_i''',\widetilde{R}_i),(\widetilde{d}',\widetilde{P})]((\widetilde{f},\widetilde{\sigma}_1),\widetilde{\sigma}_2)} \quad mod \ \overrightarrow{A}_k^{\mathbb{Z}_2}(< (r, (s_1, s_2))),$$

where $r_{\widetilde{a}}[(\widetilde{d}_i''', \widetilde{R}_i), (\widetilde{d}', \widetilde{P})]((\widetilde{f}, \widetilde{\sigma}_1), \widetilde{\sigma}_2)$ is independent of $(\widetilde{d}'', \widetilde{Q})$ and of $((\widetilde{f}, \widetilde{\sigma}_1), \widetilde{\sigma}_2)$.

*Proof.* The proof of (i)(a) and (ii)(a) follows Definitions 4.2, 4.9, 4.10 and note 2, proof of (i)(b) and (ii)(b) follows from Definition 4.11 and proof of (i)(c) and (ii)(c) follows from Lemmas 4.3, 4.4, 4.7 and 4.8. □

**Corollary 4.14.** *Let $\mathscr{A} = \mathbb{C}(x)$. A partition algebra of $\mathbb{P}_{2k}(x^2)$ is a tabular algebra together with a table datum $(\Lambda, \Gamma, B, M[(r, (s_1, s_2))], C, *)$ with $f = id$ and $s_2 = 0$.*

## 5. A Cellular Basis of the algebra of $\mathbb{Z}_2$-relations and signed partition algebras

In this section, we compute a cellular basis for the algebra of $\mathbb{Z}_2$-relations and signed partition algebras by making use of the basis defined in Lemma 4.3 and also by using cellular bases of the group algebras $\mathscr{A}[\mathbb{Z}_2 \wr \mathfrak{S}_k]$ and $\mathscr{A}[\mathfrak{S}_k]$ given in [7].

**Definition 5.1.** *Define,*

(i) $\Lambda' := \{((r, (s_1, s_2)), ((\lambda_1, \lambda_2), \mu)) \mid (r, (s_1, s_2)) \in \Lambda\}$

(ii) $\widetilde{\Lambda}' := \left\{((r, (s_1, s_2)), ((\lambda_1, \lambda_2), \mu)) \mid (r, (s_1, s_2)) \in \widetilde{\Lambda}\right\}$

*with the order given by*

$$\left(r', (s_1', s_2'), ((\lambda_1', \lambda_2'), \mu')\right) \geq \left(r, (s_1, s_2), ((\lambda_1, \lambda_2), \mu)\right)$$

*if and only if*

(a) $r' \geq r$ or

(b) $r' = r$ and $(s_1', s_2') \geq (s_1, s_2)$ i.e., $s_1' + s_2' > s_1 + s_2$

(c) $r' = r, (s_1', s_2') = (s_1, s_2)$ and $(\lambda_1', \lambda_2') \rhd (\lambda_1, \lambda_2)$.

(d) $r = r', (s_1', s_2') = (s_1, s_2), (\lambda_1', \lambda_2') = (\lambda_1, \lambda_2)$ and $\mu' \rhd \mu$.

**Definition 5.2.** *Let* $[\lambda], [\mu]$ *denote the trivial representation of* $\lambda, \mu$.

*For* $((r, (s_1, s_2)), ((\lambda_1, \lambda_2), \mu)) \in \Lambda'$ *and* $((r, (s_1, s_2)), ((\lambda_1, \lambda_2), \mu)) \in \widetilde{\Lambda}'$, *define*

$$M'[(r, (s_1, s_2)), ((\lambda_1, \lambda_2), \mu)] := M[(r, (s_1, s_2))] \times M^{((\lambda_1, \lambda_2), \mu)}$$
$$\widetilde{M}'[(r, (s_1, s_2)), ((\lambda_1, \lambda_2), \mu)] := \widetilde{M}[(r, (s_1, s_2))] \times M^{((\lambda_1, \lambda_2), \mu)}$$

*where* $M^{((\lambda_1, \lambda_2), \mu)} := \{((s_{\lambda_1}, s_{\lambda_2}), s_\mu) \mid s_{\lambda_1}, s_{\lambda_2} \text{ and } s_\mu \text{ are the standard tableaus of shape } \lambda_1, \lambda_2$
*and* $\mu$ *respectively}.*

(a) *if* $s_1 \neq 0$ *and* $s_2 \neq 0$ *then*

   (i) $M'\Big[(r, (s_1, s_2), ((\lambda_1, \lambda_2), \mu))\Big] = \Big\{\big((d', P), ((t_{\lambda_1}, t_{\lambda_2}), t_\mu)\big) \mid (d', P) \in M[(r, (s_1, s_2))],$
      $t_{\lambda_1}, t_{\lambda_2}$ *and* $t_\mu$ *are the standard tableaux of shapes* $\lambda_1, \lambda_2$ *and* $\mu$ *respectively*$\Big\}$,

   (ii) $\widetilde{M}'\Big[(r, (s_1, s_2), ((\lambda_1, \lambda_2), \mu))\Big] = \Big\{\big((\widetilde{d}', \widetilde{P}), ((t_{\lambda_1}, t_{\lambda_2}), t_\mu)\big) \mid (\widetilde{d}', \widetilde{P}) \in M[(r, (s_1, s_2))],$
      $t_{\lambda_1}, t_{\lambda_2}$ *and* $t_\mu$ *are the standard tableaux of shapes* $\lambda_1, \lambda_2$ *and* $\mu$ *respectively*$\Big\}$,

(b) *If* $s_1 \neq 0$ *and* $s_2 = 0$ *then*

   (i) $M'\Big[(r, (s_1, 0), ((\lambda_1, \lambda_2), \Phi))\Big] = \Big\{((d', P), (t_{\lambda_1}, t_{\lambda_2})) \mid (d', P) \in M[(r, (s_1, 0))], t_{\lambda_1}$
      *and* $t_{\lambda_2}$ *are the standard tableaux of shapes* $\lambda_1$ *and* $\lambda_2$ *respectively*$\Big\}$,

   (ii) $\widetilde{M}'\Big[(r, (s_1, 0), ((\lambda_1, \lambda_2), \Phi))\Big] = \Big\{((\widetilde{d}', \widetilde{P}), (t_{\lambda_1}, t_{\lambda_2})) \mid (\widetilde{d}', \widetilde{P}) \in \widetilde{M}[(r, (s_1, 0))], t_{\lambda_1}$
      *and* $t_{\lambda_2}$ *are the standard tableaux of shapes* $\lambda_1$ *and* $\lambda_2$ *respectively*$\Big\}$,

(c) *If* $s_1 = 0$ *and* $s_2 \neq 0$ *then*

   (i) $M'\Big[(r, (0, s_2), ((\Phi, \Phi), \mu))\Big] = \Big\{((d', P), t_\mu) \mid (d', P) \in M[(r, (0, s_2))], t_\mu$ *is a standard*
      *tableau of shape* $\mu$$\Big\}$,

   (ii) $\widetilde{M}'\Big[(r, (0, s_2), ((\Phi, \Phi), \mu))\Big] = \Big\{((\widetilde{d}', \widetilde{P}), t_\mu) \mid (\widetilde{d}', \widetilde{P}) \in \widetilde{M}[(r, (0, s_2))], t_\mu$ *is a standard*
      *tableau of shape* $\mu$$\Big\}$,

(d) *If* $r = 0, s_1 = 0$ *and* $s_2 = 0$ *then*

   (i) $M'\Big[(0, (0, 0), ((\Phi, \Phi), \Phi))\Big] = \Big\{(d', \Phi) \mid (d', \Phi) \in M[(0, (0, 0))]\Big\}$

   (ii) $\widetilde{M}'\Big[(0, (0, 0), ((\Phi, \Phi), \Phi))\Big] = \Big\{(\widetilde{d}', \Phi) \mid (\widetilde{d}', \Phi) \in M[(0, (0, 0))]\Big\}$

   *where* $s_1 = \natural\{C : C$ *is a connected component of* $P$ *such that* $H_C^P = \{e\}\}$ *and* $s_2 = \natural\{C : C$ *is a*
   *connected component of* $P$ *such that* $H_C^P = \mathbb{Z}_2\}$.

**Definition 5.3.** *Let*

(i) $C' : \coprod_{(r, (s_1, s_2), ((\lambda_1, \lambda_2), \mu)) \in \Lambda'} M'[(r, (s_1, s_2), ((\lambda_1, \lambda_2), \mu))] \times M'[(r, (s_1, s_2), ((\lambda_1, \lambda_2), \mu))] \to A_k^{\mathbb{Z}_2}$
   *be defined as*

$$C'[((d', P), ((s_{\lambda_1}, s_{\lambda_2}), s_\mu)), ((d'', Q), ((t_{\lambda_1}, t_{\lambda_2}), t_\mu))] = C_{(d', P), (d'', Q)}^{m_{s_\lambda, t_\lambda}^\lambda m_{s_\mu t_\mu}^\mu}$$

(ii) $\widetilde{C}' : \coprod_{(r, (s_1, s_2), ((\lambda_1, \lambda_2), \mu)) \in \widetilde{\Lambda}'} \widetilde{M}'[(r, (s_1, s_2), ((\lambda_1, \lambda_2), \mu))] \times \widetilde{M}'[(r, (s_1, s_2), ((\lambda_1, \lambda_2), \mu))] \to \overrightarrow{A}_k^{\mathbb{Z}_2}$
   *be defined as*

$$\widetilde{C}'[((\widetilde{d}', \widetilde{P}), ((s_{\lambda_1}, s_{\lambda_2}), s_\mu)), ((\widetilde{d}'', \widetilde{Q}), ((t_{\lambda_1}, t_{\lambda_2}), t_\mu))] = C_{(\widetilde{d}', \widetilde{P}), (\widetilde{d}'', \widetilde{Q})}^{m_{s_\lambda, t_\lambda}^\lambda m_{s_\mu t_\mu}^\mu}$$

*where* $m_{s_\lambda, t_\lambda}^\lambda$ *and* $m_{s_\mu t_\mu}^\mu$ *are cellular basis for the algebras* $\mathscr{A}[\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}]$ *and* $\mathscr{A}[\mathfrak{S}_{s_2}]$ *respectively.*

**Theorem 5.4.** *Let* $A_k^{\mathbb{Z}_2}\left(\overrightarrow{A}_k^{\mathbb{Z}_2}\right)$ *be the* $\mathscr{A}$*- algebra defined in Definition 2.5(2.7).*

(i) *The algebra of* $\mathbb{Z}_2$ *relations* $\mathscr{A}[A_k^{\mathbb{Z}_2}]$ *is a cellular algebra with a cell datum* $(\Lambda', M', C', *')$ *given as follows:*

(a) $\Lambda'$ *is a partially ordered set where* $\Lambda'$ *is as in Definition 5.1.*

(b) $*$ *is the unique anti involution of* $A_k^{\mathbb{Z}_2}$.

(c)    1. $aC'^{m_{s_\lambda,t_\lambda}^\lambda m_{s_\mu t_\mu}^\mu}_{(d',P),(d'',Q)} \equiv \displaystyle\sum_{S' \in M'\left[\left(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right)\right]} r_a[(d''',P'''),(d',P)]C^{'m_{s_\lambda',t_\lambda}^\lambda m_{s_\mu',t_\mu}^\mu}_{(d''',P'''),(d'',Q)}$

$$mod\ A_k^{\mathbb{Z}_2}\left(\ < \left(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right)\right),$$

*where* $r_a[(d''',P'''),(d',P)]$ *is independent of* $(d'',Q)$.

2. $aC'_{(d,\Phi),(d',\Phi)} \equiv \displaystyle\sum_{(d'',\Phi) \in M'\left[(0,(0,0),((\Phi,\Phi))\right]} r_a[(d'',\Phi),(d,\Phi)]C'_{(d'',\Phi),(d,\Phi)}.$

(ii) *The signed partition algebra is a cellular algebra* $\mathscr{A}[\overrightarrow{A}_k^{\mathbb{Z}_2}]$ *with a cell datum* $(\widetilde{\Lambda}',\widetilde{M}',\widetilde{C}',\widetilde{*}')$ *given as follows:*

(a) $\widetilde{\Lambda}'$ *is a partially ordered set where* $\widetilde{\Lambda}'$ *is as in Definition 5.1.*

(b) $\widetilde{*}$ *is the unique anti involution of* $\overrightarrow{A}_k^{\mathbb{Z}_2}$.

(c)    1. $\widetilde{a}\widetilde{C'}^{m_{s_\lambda,t_\lambda}^\lambda m_{s_\mu t_\mu}^\mu}_{(\widetilde{d}',\widetilde{P}),(\widetilde{d}'',\widetilde{Q})} \equiv \displaystyle\sum_{\widetilde{S}' \in \widetilde{M}'\left[\left(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right)\right]} r_{\widetilde{a}}[(\widetilde{d}''',\widetilde{P}'''),(\widetilde{d}',\widetilde{P})]\widetilde{C'}^{m_{s_\lambda',t_\lambda}^\lambda m_{s_\mu',t_\mu}^\mu}_{(\widetilde{d}''',\widetilde{P}'''),(\widetilde{d}'',\widetilde{Q})}$

$$mod\ \overrightarrow{A}_k^{\mathbb{Z}_2}\left(\ < \left(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right)\right),$$

*where* $r_{\widetilde{a}}[(\widetilde{d}''',\widetilde{P}'''),(\widetilde{d}',\widetilde{P})]$ *is independent of* $(\widetilde{d}'',\widetilde{Q})$.

2. $\widetilde{a}\widetilde{C}'_{(\widetilde{d},\Phi),(\widetilde{d}',\Phi)} \equiv \displaystyle\sum_{(\widetilde{d}'',\Phi) \in \widetilde{M}'\left[(0,(0,0),((\Phi,\Phi))\right]} r_{\widetilde{a}}[(\widetilde{d}'',\Phi),(\widetilde{d},\Phi)]\widetilde{C}'_{(\widetilde{d}'',\Phi),(\widetilde{d},\Phi)}.$

*Proof.* The proof follows from Theorem 4.2.1 of [3], Lemma 4.7 and Theorem 4.13. $\qquad\square$

**Remark 5.5.** *From (1.8) of* [1], $A_k^{\mathbb{Z}_2}\left(\overrightarrow{A}_k^{\mathbb{Z}_2}\right)$ *is a cellular algebra over any field* $K$ *with cell datum* $(\Lambda',M',C',*')$ $((\widetilde{\Lambda}',\widetilde{M}',\widetilde{C}',\widetilde{*}'))$ *where* $(\Lambda',M',C',*')((\widetilde{\Lambda}',\widetilde{M}',\widetilde{C}',\widetilde{*}'))$ *is as in Theorem 5.4.*

**Corollary 5.6.** *Let* $\mathbb{P}_{2k}(x^2)$ *be the* $\mathscr{A}$*- algebra defined in Definition 2.8. Then* $\mathbb{P}_{2k}(x^2)$ *has a cell datum* $(\Lambda',M',C',*')$ *with* $f=id$ *and* $s_2=0$.

## 6. Modular Representations of the algebra of $\mathbb{Z}_2$ relations and signed partition algebras

In this section, we give a description of the complete set of irreducible modules for the algebra of $\mathbb{Z}_2$ relations $A_k^{\mathbb{Z}_2}$ and signed partition algebras $\overrightarrow{A}_k^{\mathbb{Z}_2}$ over any field.

**Definition 6.1.** *Let* $r=2s_1+s_2$. *For* $0 \leq r \leq 2k$ *and* $((r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)) \in \Lambda'$ $\left(((r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)) \in \widetilde{\Lambda}'\right)$, *put* $\lambda=(\lambda_1,\lambda_2)$.

*The left cell module* $W[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]$ $\left(\widetilde{W}[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)]\right)$ *for the cellular algebra* $\mathscr{A}\left[A_k^{\mathbb{Z}_2}\right]\left(\mathscr{A}\left[\overrightarrow{A}_k^{\mathbb{Z}_2}\right]\right)$ *is defined as follows:*

(i) $W\left[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)\right]$ *is a free* $\mathscr{A}$*-module with basis*
$$\left\{ C_S^{C_{s_1,s_2}(s)} = C_S^{\prime m_{s_\lambda}^\lambda m_{s_\mu}^\mu} \;\middle|\; S=(d,P), s=((s_{\lambda_1},s_{\lambda_2}),s_\mu)\in M'\left[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)\right] \right\}$$
*and* $A_k^{\mathbb{Z}_2}$*-action is defined on the basis element by a*
$$aC_S^{\prime m_{s_\lambda}^\lambda m_{s_\mu}^\mu} \equiv \sum_{(S',s')\in M'\left[\left(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right)\right]} C_{S'}^{r_a(S',S)m_{s_\lambda'}^\lambda m_{s_\mu'}^\mu} \mod A_k^{\mathbb{Z}_2}\Big( < \left(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right) \Big),$$
*where* $(S,s)=((d,P),((s_{\lambda_1},s_{\lambda_2}),s_\mu)),(S',s')=((d',P'),((s_{\lambda_1}',s_{\lambda_2}'),s_\mu'))$ $r_a(S',S)$ *is as in 3(a)(i)*
*and (b)(i) of Theorem 5.4.*

(ii) $W\left[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)\right]$ *is a free* $\mathscr{A}$*-module with basis*
$$\left\{ \widetilde{C}_{\widetilde{S}}^{\widetilde{C}_{s_1,s_2}(s)} = \widetilde{C}_{\widetilde{S}}^{\prime m_{s_\lambda}^\lambda m_{s_\mu}^\mu} \;\middle|\; \widetilde{S}=(\widetilde{d},\widetilde{P}), s=((s_{\lambda_1},s_{\lambda_2}),s_\mu)\in \widetilde{M}'\left[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)\right] \right\}$$
*and* $\overrightarrow{A}_k^{\mathbb{Z}_2}$*-action is defined on the basis element by* $\widetilde{a}$
$$\widetilde{a}\widetilde{C}_{\widetilde{S}}^{\prime m_{s_\lambda}^\lambda m_{s_\mu}^\mu} \equiv \sum_{(\widetilde{S}',s')\in \widetilde{M}'\left[\left(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right)\right]} \widetilde{C}_{\widetilde{S}'}^{r_{\widetilde{a}}(\widetilde{S}',\widetilde{S})m_{s_\lambda'}^\lambda m_{s_\mu'}^\mu} \mod \overrightarrow{A}_k^{\mathbb{Z}_2}\Big( < \left(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right) \Big),$$
*where* $(\widetilde{S},s)=((\widetilde{d},\widetilde{P}),((s_{\lambda_1},s_{\lambda_2}),s_\mu)),(\widetilde{S}',s')=((\widetilde{d}',\widetilde{P}'),((s_{\lambda_1}',s_{\lambda_2}'),s_\mu'))$ $r_a(\widetilde{S}',\widetilde{S})$ *is as in 3(a)(ii)*
*and (b)(ii) of Theorem 5.4.*

**Lemma 6.2.**     (i) $C_{S,S}^{\prime m_{s_\lambda,s_\lambda}^\lambda m_{s_\mu,s_\mu}^\mu} \, C_{T,T}^{\prime m_{t_\lambda,t_\lambda}^\lambda m_{t_\mu,t_\mu}^\mu} \;\equiv\; \Phi_1((S,s),(T,t))\, C_{S,T}^{\prime m_{s_\lambda,t_\lambda}^\lambda m_{s_\mu,t_\mu}^\mu}$

$$\mod \left[A_k^{\mathbb{Z}_2} < (r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]$$

*where*
$$\Phi_1((S,s),(T,t)) \;=\; x^{l(P\vee P')}\phi_{\delta_1}^\lambda(s_\lambda,t_\lambda)\,\phi_{\delta_2}^\mu(s_\mu,t_\mu) \quad \textit{when conditions (a) and (b)}$$
$$\textit{of Definition 4.6 are satisfied}$$
$$= \; 0 \qquad\qquad\qquad\qquad \textit{Otherwise}$$

(ii) $\widetilde{C}_{\widetilde{S},\widetilde{S}}^{\prime m_{s_\lambda,s_\lambda}^\lambda m_{s_\mu,s_\mu}^\mu} \, \widetilde{C}_{\widetilde{T},\widetilde{T}}^{\prime m_{t_\lambda,t_\lambda}^\lambda m_{t_\mu,t_\mu}^\mu} \;\equiv\; \Phi_1((\widetilde{S},s),(\widetilde{T},t)) \widetilde{C}_{\widetilde{S},\widetilde{T}}^{\prime m_{s_\lambda,t_\lambda}^\lambda m_{s_\mu,t_\mu}^\mu} \qquad \mod \left[\overrightarrow{A}_k^{\mathbb{Z}_2} < (r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]$

*where*
$$\Phi_1((\widetilde{S},s),(\widetilde{T},t)) \;=\; x^{l(\widetilde{P}\vee \widetilde{P}')}\phi_{\delta_1}^\lambda(s_\lambda,t_\lambda)\,\phi_{\delta_2}^\mu(s_\mu,t_\mu) \quad \textit{when conditions (a) and (b)}$$
$$\textit{of Definition 4.6 are satisfied}$$
$$= \; 0 \qquad\qquad\qquad\qquad \textit{Otherwise}$$

$(S,s) = ((d,P),((s_{\lambda_1},s_{\lambda_2}),s_\mu)),(\widetilde{S},s) = ((\widetilde{d},\widetilde{P}),((s_{\lambda_1},s_{\lambda_2}),s_\mu)),(T,t) = ((d',P'),((t_{\lambda_1},t_{\lambda_2}),t_\mu)),(\widetilde{T},t) =$ $((\widetilde{d}',\widetilde{P}'),((t_{\lambda_1},t_{\lambda_2}),t_\mu)),l(P\vee P')\left(l(\widetilde{P}\vee\widetilde{P}')\right)$ *denotes the number of*
*connected components in* $d'.d''$ $\left(\widetilde{d}'.\widetilde{d}''\right)$ *excluding the union of all the connected components of* $P$ *and* $P'$ $\left(\widetilde{P}$ *and* $\widetilde{P}'\right)$,
$m_{s_\lambda,s_\lambda}^\lambda \delta_1 m_{t_\lambda,t_\lambda}^\lambda \equiv \phi_{\delta_1}^\lambda(s_\lambda,t_\lambda)m_{s_\lambda',t_\lambda}^\lambda \mod \mathscr{H}\left(<(\lambda_1,\lambda_2)\right), m_{s_\mu,s_\mu}^\mu \delta_2 m_{t_\mu,t_\mu}^\mu \equiv \phi_{\delta_2}^\mu(s_\mu,t_\mu)m_{s_\mu',t_\mu}^\mu \mod \mathscr{H}'\left(<\mu\right)$
*as in Lemma 1.7* [1].

*Proof.* **Proof of (i):** Consider the product
$$C_{S,S}^{\prime m_{s_\lambda,s_\lambda}^\lambda m_{s_\mu,s_\mu}^\mu} \, C_{T,T}^{\prime m_{t_\lambda,t_\lambda}^\lambda m_{t_\mu,t_\mu}^\mu} \;=\; x^{l(P\vee P')}C_{S,T}^{\prime m_{s_\lambda,s_\lambda}^\lambda m_{s_\mu,s_\mu}^\mu(\delta_1,\delta_2)m_{t_\lambda,t_\lambda}^\lambda m_{t_\mu,t_\mu}^\mu}$$

where $\phi_{s_1,s_2}^r((d,P),(d',P')) = x^{l(P\vee P')}(\delta_1,\delta_2)$ is as in Definition 4.6,
We know that,

$$m^\lambda_{s_\lambda,s_\lambda} m^\mu_{s_\mu,s_\mu} (\delta_1, \delta_2) m^\lambda_{t_\lambda,t_\lambda} m^\mu_{t_\mu,t_\mu} \;=\; m^\lambda_{s_\lambda,s_\lambda} \delta_1 m^\lambda_{t_\lambda,t_\lambda} m^\mu_{s_\mu,s_\mu} \delta_2 m^\mu_{t_\mu,t_\mu} \tag{6.1}$$

$$=\; \phi^\lambda_{\delta_1}(s_\lambda, t_\lambda) m^\lambda_{s'_\lambda,t_\lambda} \phi^\mu_{\delta_2}(s_\mu, t_\mu) m^\mu_{s'_\mu,t_\mu}$$

$$=\; \phi^\lambda_{\delta_1}(s_\lambda, t_\lambda)\ \phi^\mu_{\delta_2}(s_\mu, t_\mu)\ m^\lambda_{s'_\lambda,t_\lambda} m^\mu_{s'_\mu,t_\mu}$$

where $m^\lambda_{s_\lambda,s_\lambda} \delta_1 m^\lambda_{t_\lambda,t_\lambda} \equiv \phi^\lambda_{\delta_1}(s_\lambda, t_\lambda) m^\lambda_{s'_\lambda,t_\lambda} \mod \mathscr{H}\left(< (\lambda_1, \lambda_2)\right)$,

$m^\mu_{s_\mu,s_\mu} \delta_2 m^\mu_{t_\mu,t_\mu} \equiv \phi^\mu_{\delta_2}(s_\mu, t_\mu) m^\mu_{s'_\mu,t_\mu} \mod \mathscr{H}'\left(< \mu\right)$.

Substitute the above in the product $C'^{m^\lambda_{s_\lambda,s_\lambda} m^\mu_{s_\mu,s_\mu}}_{S,S}\ C'^{m^\lambda_{t_\lambda,t_\lambda} m^\mu_{t_\mu,t_\mu}}_{T,T}$ we get,

$$C'^{m^\lambda_{s_\lambda,s_\lambda} m^\mu_{s_\mu,s_\mu}}_{S,S}\ C'^{m^\lambda_{t_\lambda,t_\lambda} m^\mu_{t_\mu,t_\mu}}_{T,T} \;=\; x^{l(P\vee P')}\phi^\lambda_{\delta_1}(s_\lambda, t_\lambda)\ \phi^\mu_{\delta_2}(s_\mu, t_\mu) C'^{m^\lambda_{s'_\lambda,t_\lambda} m^\mu_{s'_\mu,t_\mu}}_{S,T}$$

$$=\; \Phi_1((S,s),(T,t))\ C^{m^\lambda_{s'_\lambda,t_\lambda} m^\mu_{s'_\mu,t_\mu}}_{S,T}$$

where $\Phi_1((S,s),(T,t)) = x^{l(P\vee P')}\phi^\lambda_{\delta_1}(s_\lambda, t_\lambda)\ \phi^\mu_{\delta_2}(s_\mu, t_\mu)$.

**Proof of (ii):** Proof of (ii) is same as proof of (i). $\qquad\qquad\qquad\square$

**Definition 6.3.** *For* $\left(r, (s_1, s_2), ((\lambda_1, \lambda_2), \mu)\right) \in \Lambda'\ \left((r, (s_1, s_2), ((\lambda_1, \lambda_2), \mu)) \in \widetilde{\Lambda}'\right)$, *the bilinear map* $\phi^{\lambda,\mu}_{s_1,s_2}$ $\left(\widetilde{\phi}^{\lambda,\mu}_{s_1,s_2}\right)$ *is defined as*

(i) $\phi^{\lambda,\mu}_{s_1,s_2}\big(C'^{m^\lambda_{s_\lambda,s_\lambda} m^\mu_{s_\mu,s_\mu}}_{(d,P)}, C'^{m^\lambda_{t_\lambda,t_\lambda} m^\mu_{t_\mu,t_\mu}}_{(d',P')}\big) = \Phi_1((S,s),(T,t)),\quad (S,s),(T,t) \in M'\big[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\big]$

(ii) $\widetilde{\phi}^{\lambda,\mu}_{s_1,s_2}\big(\widetilde{C}'^{m^\lambda_{s_\lambda,s_\lambda} m^\mu_{s_\mu,s_\mu}}_{(\widetilde{d},\widetilde{P})}, \widetilde{C}'^{m^\lambda_{t_\lambda,t_\lambda} m^\mu_{t_\mu,t_\mu}}_{(\widetilde{d}',\widetilde{P}')}\big) = \Phi_1((\widetilde{S},s),(\widetilde{T},t)),\quad (\widetilde{S},s),(\widetilde{T},t) \in \widetilde{M}'\big[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\big]$

*where* $\Phi_1((S,s),(T,t)) \left(\widetilde{\Phi}_1((\widetilde{S},s),(\widetilde{T},t))\right)$ *is as in Lemma 6.2.*

*Put*

(i) $G^{\lambda,\mu}_{2s_1+s_2} = \left(\Phi_1((S,s),(T,t))\right)_{(S,s),(T,t)\in M'\big[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\big]}$

    *where*

$$\Phi_1((S,s),(T,t)) \;=\; x^{l(P_i\vee P_j)}\phi^\lambda_{\delta_1}(s_\lambda, t_\lambda)\ \phi^\mu_{\delta_2}(s_\mu, t_\mu) \quad \text{when conditions (a) and (b)}$$
$$\text{of Definition 4.6 are satisfied}$$
$$=\; 0 \qquad\qquad\qquad\qquad \text{Otherwise}$$

    *where* $(S,s) = ((d_i, P_i), ((s_{\lambda_1}, s_{\lambda_2}), s_\mu)), (T,t) = ((d_j, P_j), ((t_{\lambda_1}, t_{\lambda_2}), t_\mu))$

(ii) $\widetilde{G}^{\lambda,\mu}_{2s_1+s_2} = \left(\widetilde{\Phi}_1((\widetilde{S},s),(\widetilde{T},t))\right)_{(\widetilde{S},s),(\widetilde{T},t)\in \widetilde{M}'\big[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\big]}$

    *where*

$$\widetilde{\Phi}_1((\widetilde{S},s),(\widetilde{T},t)) \;=\; x^{l(\widetilde{P}_i\vee \widetilde{P}_j)}\phi^\lambda_{\delta_1}(s_\lambda, t_\lambda)\ \phi^\mu_{\delta_2}(s_\mu, t_\mu) \quad \text{when conditions (a) and (b)}$$
$$\text{of Definition 4.6 are satisfied}$$
$$=\; 0 \qquad\qquad\qquad\qquad \text{Otherwise}$$

    *where* $(\widetilde{S},s) = ((\widetilde{d}_i, \widetilde{P}_i), ((s_{\lambda_1}, s_{\lambda_2}), s_\mu)), (\widetilde{T},t) = ((\widetilde{d}_j, \widetilde{P}_j), ((t_{\lambda_1}, t_{\lambda_2}), t_\mu))$,

$l(P_i \vee P_j) \left(l(\widetilde{P}_i \vee \widetilde{P}_j)\right)$ *denotes the number of connected components in* $d'.d'' \left(\widetilde{d}'.\widetilde{d}''\right)$ *excluding the union of all the connected components of* $P_i$ *and* $P_j$ $\left(\widetilde{P}_i \text{ and } \widetilde{P}_j\right)$,

$m^\lambda_{s_\lambda,s_\lambda} \delta_1 m^\lambda_{t_\lambda,t_\lambda} \equiv \phi^\lambda_{\delta_1}(s_\lambda, t_\lambda) m^\lambda_{s'_\lambda,t_\lambda} \mod \mathscr{H}\left(< (\lambda_1, \lambda_2)\right), m^\mu_{s_\mu,s_\mu} \delta_2 m^\mu_{t_\mu,t_\mu} \equiv \phi^\mu_{\delta_2}(s_\mu, t_\mu) m^\mu_{s'_\mu,t_\mu} \mod \mathscr{H}'\left(< \mu\right)$

*as in Lemma 1.7* [1].

$G_{2s_1+s_2}^{\lambda,\mu}\left(\widetilde{G}_{2s_1+s_2}^{\lambda,\mu}\right)$ *is called the **Gram matrix of the cell module*** $W\left[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)\right]$
$\left(\widetilde{W}\left[(r,(s_1,s_2)),((\lambda_1,\lambda_2),\mu)\right]\right).$

**Definition 6.4.** *For* $\left(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right) \in \Lambda'\left(\left(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right) \in \widetilde{\Lambda}'\right)$, *define*

(i) $Rad\left(W\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]\right) = \left\{x \in W\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right] \mid \right.$
$$\phi_{s_1,s_2}^{\lambda,\mu}(x,y) = 0 \ \ \forall \ y \in W\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]\right\},$$

(ii) $Rad\left(\widetilde{W}\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]\right) = \left\{x \in \widetilde{W}\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right] \mid \right.$
$$\widetilde{\phi}_{s_1,s_2}^{\lambda,\mu}(x,y) = 0 \ \ \forall \ y \in \widetilde{W}\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]\right\},$$

*where* $(S,s) = ((d,P),((s_{\lambda_1},s_{\lambda_2}),s_\mu)), (\widetilde{S},s) = ((\widetilde{d},\widetilde{P}),((s_{\lambda_1},s_{\lambda_2}),s_\mu)), (T,t) = ((d',P'),((t_{\lambda_1},t_{\lambda_2}),t_\mu))$ *and*
$(\widetilde{T},t) = ((\widetilde{d'},\widetilde{P'}),((t_{\lambda_1},t_{\lambda_2}),t_\mu)).$

**Notation 6.5.** *Let*

(i) $\Lambda_0' = \{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)) \in \Lambda' \mid \phi_{s_1,s_2}^{\lambda,\mu} \neq 0\}.$

(ii) $\widetilde{\Lambda}_0' = \{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)) \in \widetilde{\Lambda}' \mid \widetilde{\phi}_{s_1,s_2}^{\lambda,\mu} \neq 0\}.$

**Theorem 6.6.** *Let* $\mathbb{K}(x)$ *be a field. For* $(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)) \in \Lambda_0'\left((r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)) \in \widetilde{\Lambda}_0'\right),$
*let*

(i) $D^{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)} = \dfrac{W\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]}{Rad\left(W\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]\right)},$

(ii) $\widetilde{D}^{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)} = \dfrac{\widetilde{W}\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]}{Rad\left(\widetilde{W}\left[r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)\right]\right)}.$

(a) $D^{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)} \neq 0 \left(\widetilde{D}^{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)} \neq 0\right)$ *if and only if* $\lambda = (\lambda_1,\lambda_2)$ *is p- restricted and* $\mu$ *is p- restricted and it is absolutely irreducible over a field of characteristic p.*

(a)′ $D^{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)} \neq 0 \left(\widetilde{D}^{(r,(s_1,s_2),((\lambda_1,\lambda_2),\mu)} \neq 0\right)$ *and it is absolutely irreducible over a field of characteristic 0.*

(b) $D^{(r,(s_1,0),(\lambda_1,\lambda_2)} \neq 0 \left(\widetilde{D}^{(r,(s_1,0),(\lambda_1,\lambda_2)} \neq 0\right)$ *if and only if* $\lambda = (\lambda_1,\lambda_2)$ *is p- restricted and it is absolutely irreducible over a field of characteristic p.*

(b)′ $D^{(r,(s_1,0),(\lambda_1,\lambda_2)} \neq 0 \left(\widetilde{D}^{(r,(s_1,0),(\lambda_1,\lambda_2)} \neq 0\right)$ *and it is absolutely irreducible over a field of characteristic 0.*

(c) $D^{(r,(0,s_2),\mu)} \neq 0 \left(\widetilde{D}^{(r,(0,s_2),\mu)} \neq 0\right)$ *if and only if* $\mu$ *is p- restricted and it is absolutely irreducible over a field of characteristic p.*

(c)′ $D^{(r,(0,s_2),\mu)} \neq 0 \left(\widetilde{D}^{(r,(0,s_2),\mu)} \neq 0\right)$ *and it is absolutely irreducible over a field of characteristic 0.*

(d) $D^{(0,\Phi)}\left(\widetilde{D}^{(0,\Phi)}\right)$ *is non-zero and it is absolutely irreducible over a field of characteristic 0.*

*Proof.* We shall show that $\Phi_1((S,s),(T,t)) \neq 0$ for some $(S,s),(T,t)$.

Consider $(S,s) = ((d,P),((s_{\lambda_1},s_{\lambda_2}),s_\mu))$ and $(S',s') = ((d,P),((s_{\lambda_1}',s_{\lambda_2}'),s_\mu'))$ then

$$\Phi_1((S,s),(S',s')) = x^{l(P\vee P)}\phi_1(s_\lambda,s_\lambda')\phi_1(s_\mu,s_\mu'),$$

where $\lambda = (\lambda_1, \lambda_2), \phi_1(s_\lambda, s'_\lambda)$ and $\phi_1(s_\mu, s'_\mu)$ are the bilinear forms of the cell module $W^\lambda$ and $W^\mu$ of the cellular algebras $k[\mathbb{Z}_2 \wr \mathfrak{S}_{s_1}]$ and $K[\mathfrak{S}_{s_2}]$ respectively.

We know that $\phi_1(s_\lambda, s'_\lambda) \neq 0$ and $\phi_1(s_\mu, s_\mu) \neq 0$ for some $s'_\lambda$ and $s'_\mu$ which implies that

$$\Phi_1((S, s), (T, t)) \neq 0$$

for some $(S, s) = \big((d, P), ((s_{\lambda_1}, s_{\lambda_2}), s_\mu)\big), (T, t) = \big((d, Q), ((t_{\lambda_1}, t_{\lambda_2}), t_\mu)\big)$.

Conversely, assume that $\Phi_1((S, s), (T, t)) \neq 0 \qquad$ for some $(S, s), (T, t)$.

$$i.e., \Phi_1((S, s), (T, t)) = x^{l(P \vee Q)} \phi^\lambda_{\delta_1}(s_\lambda, t_\lambda) \phi^\mu_{\delta_2}(s_\mu, t_\mu) \neq 0$$

which implies that

$$\phi^\lambda_{\delta_1}(s_\lambda, t_\lambda) \neq 0, \phi^\mu_{\delta_2}(s_\mu, t_\mu) \neq 0 \tag{6.2}$$

where $\phi^r_{s_1, s_2}((d, P), (d', Q)) = x^{l(P \vee Q)}(\delta_1, \delta_2)$ is as in Definition 4.6,

$m^\lambda_{s_\lambda, s_\lambda} \delta_1 m^\lambda_{t_\lambda, t_\lambda} \equiv \phi^\lambda_{\delta_1}(s_\lambda, t_\lambda) m^\lambda_{s_\lambda, t_\lambda} \mod \mathscr{H}(< (\lambda_1, \lambda_2))$ and

$m^\mu_{s_\mu, s_\mu} \delta_2 m^\mu_{t_\mu, t_\mu} \equiv \phi^\mu_{\delta_2}(s_\mu, t_\mu) m^\mu_{s_\mu, t_\mu} \mod \mathscr{H}'(< \mu)$.

Also we know that by proof of (ii) of proposition 2.4 in [1],

$$\phi^\lambda_{\delta_1}(s_\lambda, t_\lambda) = \sum r_{\delta^\lambda_1}(s'_\lambda, t_\lambda) \phi_1(s_\lambda, t_\lambda) \text{ and } \phi^\mu_{\delta_2}(s_\mu, t_\mu) = \sum r_{\delta^\mu_2}(s'_\mu, t_\mu) \phi_1(s_\mu, t_\mu)$$

By equation (6.2) we have,

$\phi_1(s_\lambda, t_\lambda) \neq 0$ and $\phi_1(s_\mu, t_\mu) \neq 0$ for some $t_\lambda$ and $t_\mu$.

Thus the proof of (a), (b), (c) follows from [7] and (7.6) of [6] and the absolute irreducibility follows Proposition 3.2 of [1]. $\qquad\qquad\square$

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